Riemann Stieltjes Integrals

➢ *Introduction*

In elementary treatment of Integral Calculus the subject of integration is treated as inverse of differentiation. The subject arose in connection with the determination of areas of plane regions and was based on the notion of the limit of a type of sum when the number of terms in the sum tends to infinity and each term tends to zero. In fact the name Integral Calculus has its origin in this process of summation. It was only afterwards that it was seen that the subject of integration can also be viewed from the point of the inverse of differentiation.

➢ *Partition*

Let $[a,b]$ be a given interval. A finite set $P=\{a=x_0,x_1,x_2,....,x_k,....,x_n=b\}$ is said to be a partition of $[a,b]$ which divides it into n such intervals

$$[x_0,x_1],[x_1,x_2],[x_2,x_3],......,[x_{n-1},x_n]$$

Each sub-interval is called a *component* of the partition.

Obviously, corresponding to different choices of the points x_i we shall have different partition.

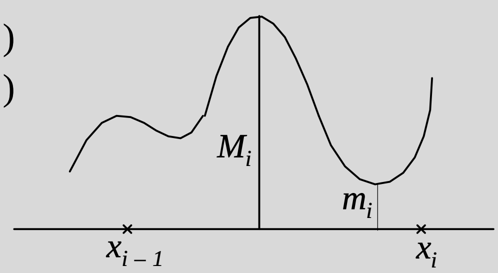
The maximum of the length of the components is defined as the *norm* of the partition.

➢ *Riemann Integral*

Let f be a real-valued function defined and bounded on $[a,b]$. Corresponding to each partition P of $[a,b]$, we put

$$M_i=\sup f(x) \qquad (x_{i-1}\le x\le x_i)$$

$$m_i=\inf f(x) \qquad (x_{i-1}\le x\le x_i)$$



We define upper and lower sums as

$$U(P,f)=\sum_{i=1}^{n}M_i\,\Delta x_i$$

and $$L(P,f)=\sum_{i=1}^{n}m_i\,\Delta x_i$$

where $\Delta x_i=x_i-x_{i-1} \quad (i=1,2,....,n)$

and finally $$\overline{\int_a^b} f\,dx=\inf U(P,f) \text{ .... } (i)$$

$$\underline{\int_a^b} f\,dx=\sup L(P,f) \text{ .... } (ii)$$

Where the infimum and the supremum are taken over all partitions P of $[a,b]$.

Then $\overline{\int_a^b} f\,dx$ and $\underline{\int_a^b} f\,dx$ are called the upper and lower Riemann Integrals of f over $[a,b]$ respectively.

In case the upper and lower integrals are equal, we say that f is Riemann-Integrable on $[a,b]$ and we write $f\in \boldsymbol{R}$, where $\boldsymbol{R}$ denotes the set of Riemann integrable functions.

The common value of (i) and (ii) is denoted by $\int_a^b f\,dx$ or by $\int_a^b f(x)\,dx$.

Which is known as the Riemann integral of f over $[a,b]$.

➢ *Theorem*

The upper and lower integrals are defined for every bounded function f.

Proof

Take M and m to be the upper and lower bounds of $f(x)$ in $[a,b]$.

$$\Rightarrow \quad m \le f(x) \le M \qquad (a \le x \le b)$$

Then $M_i \le M$ and $m_i \ge m$ $\qquad (i = 1,2,.....,n)$

Where M_i and m_i denote the supremum and infimum of $f(x)$ in $\left(x_{i-1}, x_i\right)$ for certain partition P of $\left[a,b\right]$.

$$\Rightarrow L\left(P, f\right) = \sum_{i=1}^{n} m_i\,\Delta x_i \ge \sum_{i=1}^{n} m\,\Delta x_i \qquad (\Delta x_i = x_{i-1} - x_i)$$

$$\Rightarrow L\left(P, f\right) \ge m\sum_{i=1}^{n} \Delta x_i$$

But $\displaystyle\sum_{i=1}^{n} \Delta x_i = (x_1 - x_0) + (x_2 - x_1) + (x_3 - x_2) + ... + (x_n - x_{n-1})$

$$= x_n - x_0 = b - a$$

$$\Rightarrow L\left(P, f\right) \ge m(b-a)$$

Similarity $U\left(P, f\right) \le M(b-a)$

$$\Rightarrow m(b-a) \le L\left(P, f\right) \le U\left(P, f\right) \le M(b-a)$$

Which shows that the numbers $L\left(P, f\right)$ and $U\left(P, f\right)$ form a bounded set.

$\Rightarrow$ The upper and lower integrals are defined for every bounded function f. ⊙

➢ *Riemann-Stieltjes Integral*

It is a generalization of the Riemann Integral. Let $\alpha(x)$ be a monotonically increasing function on $[a,b]$. $\alpha(a)$ and $\alpha(b)$ being finite, it follows that $\alpha(x)$ is bounded on $[a,b]$. Corresponding to each partition P of $[a,b]$, we write

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$$

(Difference of values of α at x_i & x_{i-1})

$\because$ $\alpha(x)$ is monotonically increasing.

$\therefore$ $\Delta\alpha_i \ge 0$

Let f be a real function which is bounded on $[a,b]$.

Put
$$U\left(P, f, \alpha\right) = \sum_{i=1}^{n} M_i\,\Delta\alpha_i$$

$$L\left(P, f, \alpha\right) = \sum_{i=1}^{n} m_i\,\Delta\alpha_i$$

Where M_i and m_i have their usual meanings.

Define

$$\int_a^{\bar{b}} f\,d\alpha = \inf U\left(P, f, \alpha\right) \ .... \ (i)$$

$$\int_{\underline{a}}^{b} f\,d\alpha = \sup L\left(P, f, \alpha\right) \ .... \ (ii)$$

Where the infimum and supremum are taken over all partitions of $[a,b]$.

If $\overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha$, we denote their common value by $\int_a^b f\,d\alpha$ or $\int_a^b f(x)\,d\alpha(x)$.

This is the Riemann-Stieltjes integral or simply the Stieltjes Integral of f w.r.t. α over $[a,b]$.

If $\int_a^b f\,d\alpha$ exists, we say that f is integrable w.r.t. α, in the Riemann sense, and write $f \in \boldsymbol{R}(\alpha)$.

➢ Note

The Riemann-integral is a special case of the Riemann-Stieltjes integral when we take $\alpha(x) = x$.

$\because$ The integral depends upon f, α, a and b but not on the variable of integration.

$\therefore$ We can omit the variable and prefer to write $\int_a^b f\,d\alpha$ instead of $\int_a^b f(x)\,d\alpha(x)$.

In the following discussion f will be assume to be real and bounded, and α monotonically increasing on $[a,b]$.

➢ Refinement of a Partition

Let P and P^* be two partitions of an interval $[a,b]$ such that $P \subset P^*$ i.e. every point of P is a point of P^*, then P^* is said to be a *refinement* of P.

➢ Common Refinement

Let P_1 and P_2 be two partitions of $[a,b]$. Then a partition P^* is said to be their *common refinement* if $P^* = P_1 \cup P_2$.

➢ Theorem

If P^* is a refinement of P, then

$$L(P, f, \alpha) \le L(P^*, f, \alpha) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ (i)$$

and $$U(P, f, \alpha) \ge U(P^*, f, \alpha) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ (ii)$$

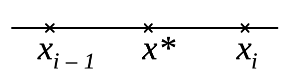
Proof

Let us suppose that P^* contains just one point x^* more than P such that $x_{i-1} < x^* < x_i$ where x_{i-1} and x_i are two consecutive points of P.

Put

$$w_1 = \inf f(x) \qquad (x_{i-1} \le x \le x^*)$$

$$w_2 = \inf f(x) \qquad (x^* \le x \le x_i)$$



It is clear that $w_1 \ge m_i$ & $w_2 \ge m_i$ where $m_i = \inf f(x)$, $(x_{i-1} \le x \le x_i)$.

Hence

$$\begin{aligned} L(P^*, f, \alpha) - L(P, f, \alpha) &= w_1\left[\alpha(x^*) - \alpha(x_{i-1})\right] + w_2\left[\alpha(x_i) - \alpha(x^*)\right] \\ &\quad - m_i\left[\alpha(x_i) - \alpha(x_{i-1})\right] \\ &= w_1\left[\alpha(x^*) - \alpha(x_{i-1})\right] + w_2\left[\alpha(x_i) - \alpha(x^*)\right] \\ &\quad - m_i\left[\alpha(x_i) - \alpha(x^*) + \alpha(x^*) - \alpha(x_{i-1})\right] \\ &= (w_1 - m_i)\left[\alpha(x^*) - \alpha(x_{i-1})\right] + (w_2 - m_i)\left[\alpha(x_i) - \alpha(x^*)\right] \end{aligned}$$

$\because$ α is a monotonically increasing function.

$\therefore$ $\alpha(x^*)-\alpha(x_{i-1})\geq 0$, $\alpha(x_i)-\alpha(x^*)\geq 0$

$$\Rightarrow L\left(P^*,f,\alpha\right)-L\left(P,f,\alpha\right) \geq 0$$

$$\Rightarrow L\left(P,f,\alpha\right) \leq L\left(P^*,f,\alpha\right) \qquad \text{which is } (i)$$

If P^* contains k points more than P, we repeat this reasoning k times and arrive at (i).

Now put

$$W_1=\sup f(x) \qquad (x_{i-1}\leq x\leq x^*)$$

and $$W_2=\sup f(x) \qquad (x^*\leq x\leq x_i)$$

Clearly $M_i\geq W_1$ & $M_i\geq W_2$

Consider

$$\begin{aligned}U\left(P,f,\alpha\right)-U\left(P^*,f,\alpha\right)&=M_i\left[\alpha(x_i)-\alpha(x_{i-1})\right]\\&\quad -W_1\left[\alpha(x^*)-\alpha(x_{i-1})\right]-W_2\left[\alpha(x_i)-\alpha(x^*)\right]\\&=M_i\left[\alpha(x_i)-\alpha(x^*)+\alpha(x^*)-\alpha(x_{i-1})\right]\\&\quad -W_1\left[\alpha(x^*)-\alpha(x_{i-1})\right]-W_2\left[\alpha(x_i)-\alpha(x^*)\right]\\&=\left(M_i-W_1\right)\left[\alpha(x^*)-\alpha(x_{i-1})\right]+\left(M_i-W_2\right)\left[\alpha(x_i)-\alpha(x^*)\right] \geq 0\\&\qquad\qquad\left(\because \alpha \text{ is } \uparrow\right)\end{aligned}$$

$$\Rightarrow U\left(P,f,\alpha\right) \geq U\left(P^*,f,\alpha\right) \qquad \text{which is } (ii)$$

⊙

➢ *Theorem*

Let f be a real valued function defined on $[a,b]$ and α be a monotonically increasing function on $[a,b]$. Then

$$\sup L\left(P,f,\alpha\right) \leq \inf U\left(P,f,\alpha\right)$$

$$\text{i.e.} \quad \underline{\int_a^b} f\,d\alpha \leq \overline{\int_a^b} f\,d\alpha$$

Proof

Let P^* be the common refinement of two partitions P_1 and P_2. Then

$$L\left(P_1,f,\alpha\right) \leq L\left(P^*,f,\alpha\right) \leq U\left(P^*,f,\alpha\right) \leq U\left(P_2,f,\alpha\right)$$

Hence $L\left(P_1,f,\alpha\right) \leq U\left(P_2,f,\alpha\right)$ …………. (i)

If P_2 is fixed and the supremum is taken over all P_1 then (i) gives

$$\underline{\int_a^b} f\,d\alpha \leq U\left(P_2,f,\alpha\right)$$

Now take the infimum over all P_2

$$\Rightarrow \underline{\int_a^b} f\,d\alpha \leq \overline{\int_a^b} f\,d\alpha$$ ⊙

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> **➢ *Theorem (Condition of Integrability or Cauchy's Criterion for Integrability.)***
> $f \in \boldsymbol{R}(\alpha)$ on $[a,b]$ iff for every $\varepsilon > 0$ there exists a partition P such that
> $$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$$

Proof

Let $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$ ………… (i)

Then $L(P,f,\alpha) \le \underline{\int_a^b} f\,d\alpha \le \overline{\int_a^b} f\,d\alpha \le U(P,f,\alpha)$

$$\Rightarrow \underline{\int_a^b} f\,d\alpha - L(P,f,\alpha) \ge 0 \quad \text{and} \quad U(P,f,\alpha) - \overline{\int_a^b} f\,d\alpha \ge 0$$

Adding these two results, we have

$$\underline{\int_a^b} f\,d\alpha - \overline{\int_a^b} f\,d\alpha - L(P,f,\alpha) + U(P,f,\alpha) \ge 0$$

$$\Rightarrow \overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha \le U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon \qquad \text{from } (i)$$

i.e. $0 \le \overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha < \varepsilon$ for every $\varepsilon > 0$.

$$\Rightarrow \overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha \qquad \text{i.e.} \quad f \in \boldsymbol{R}(\alpha)$$

Conversely, let $f \in \boldsymbol{R}(\alpha)$ and let $\varepsilon > 0$

$$\Rightarrow \overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha = \int_a^b f\,d\alpha$$

Now $\overline{\int_a^b} f\,d\alpha = \inf U(P,f,\alpha)$ and $\underline{\int_a^b} f\,d\alpha = \sup L(P,f,\alpha)$

There exist partitions P_1 and P_2 such that

$$U(P_2,f,\alpha) - \int_a^b f\,d\alpha < \frac{\varepsilon}{2} \quad \text{…………… } (ii)$$

and $\int_a^b f\,d\alpha - L(P_1,f,\alpha) < \frac{\varepsilon}{2}$ …………… (iii)

$U(P_2,f,\alpha) - \varepsilon/2 < \int f\,d\alpha$

$\int f\,d\alpha < L(P_1,f,\alpha) + \varepsilon/2$

We choose P to be the common refinement of P_1 and P_2.

Then

$$U(P,f,\alpha) \le U(P_2,f,\alpha) < \int_a^b f\,d\alpha + \frac{\varepsilon}{2} < L(P_1,f,\alpha) + \varepsilon \le L(P,f,\alpha) + \varepsilon$$

So that

$$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$$

⊙

> ***Theorem***
>
> ***a***) If $U(P,f,\alpha)-L(P,f,\alpha)<\varepsilon$ holds for some P and some ε, then it holds (with the same ε) for every refinement of P.
>
> ***b***) If $U(P,f,\alpha)-L(P,f,\alpha)<\varepsilon$ holds for $P=\{x_0,....,x_n\}$ and s_i,t_i are arbitrary points in $[x_{i-1},x_i]$, then
> $$\sum_{i=1}^{n}|f(s_i)-f(t_i)|\Delta\alpha_i<\varepsilon$$
>
> ***c***) If $f\in\boldsymbol{R}(\alpha)$ and the hypotheses of (*b*) holds, then
> $$\left|\sum_{i=1}^{n}f(t_i)\Delta\alpha_i-\int_a^b f\,d\alpha\right|<\varepsilon$$

Proof

a) Let P^* be a refinement of P. Then

$$L(P,f,\alpha)\ \le\ L(P^*,f,\alpha)$$

$$\text{and}\quad U(P^*,f,\alpha)\ \le\ U(P,f,\alpha)$$

$$\Rightarrow\ L(P,f,\alpha)+U(P^*,f,\alpha)\ \le\ L(P^*,f,\alpha)+U(P,f,\alpha)$$

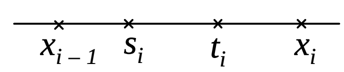
$$\Rightarrow\ U(P^*,f,\alpha)-L(P^*,f,\alpha)\ \le\ U(P,f,\alpha)-L(P,f,\alpha)$$

$$\because\ U(P,f,\alpha)-L(P,f,\alpha)<\varepsilon$$

$$\therefore\ U(P^*,f,\alpha)-L(P^*,f,\alpha)<\varepsilon$$

b) $P=\{x_0,....,x_n\}$ and s_i, t_i are arbitrary points in $[x_{i-1},x_i]$.

$$\Rightarrow\ f(s_i)\text{ and }f(t_i)\text{ both lie in }[m_i,M_i].$$



$$\Rightarrow\ |f(s_i)-f(t_i)|\ \le\ M_i-m_i$$

$$\Rightarrow\ |f(s_i)-f(t_i)|\Delta\alpha_i\ \le\ M_i\Delta\alpha_i-m_i\Delta\alpha_i$$

$$\Rightarrow\ \sum_{i=1}^{n}|f(s_i)-f(t_i)|\Delta\alpha_i\ \le\ \sum_{i=1}^{n}M_i\,\Delta\alpha_i-\sum_{i=1}^{n}m_i\,\Delta\alpha_i$$

$$\Rightarrow\ \sum_{i=1}^{n}|f(s_i)-f(t_i)|\Delta\alpha_i\ \le\ U(P,f,\alpha)-L(P,f,\alpha)$$

$$\because\ U(P,f,\alpha)-L(P,f,\alpha)\ <\ \varepsilon$$

$$\therefore\ \sum_{i=1}^{n}|f(s_i)-f(t_i)|\Delta\alpha_i\ <\ \varepsilon$$

c) $\quad\because\ m_i\ \le\ f(t_i)\ \le\ M_i$

$$\therefore\ \sum m_i\,\Delta\alpha_i\ \le\ \sum f(t_i)\Delta\alpha_i\ \le\ \sum M_i\,\Delta\alpha_i$$

$$\Rightarrow\ L(P,f,\alpha)\ \le\ \sum f(t_i)\Delta\alpha_i\ \le\ U(P,f,\alpha)$$

and also $\quad L(P,f,\alpha)\ \le\ \int_a^b f\,d\alpha\ \le\ U(P,f,\alpha)$

Using (*b*), we have

$$\left|\sum f(t_i)\Delta\alpha_i-\int_a^b f\,d\alpha\right|\ <\ \varepsilon$$

⊙

> **Theorem**
> If f is continuous on $[a,b]$ then $f \in \boldsymbol{R}(\alpha)$ on $[a,b]$.

Proof

Let $\varepsilon > 0$ be given. Choose $\beta > 0$ so that

$$\left[\alpha(b)-\alpha(a)\right]\beta < \varepsilon$$

f is continuous on $[a,b]$ $\Rightarrow$ f is uniformly continuous on $[a,b]$.

$\Rightarrow$ There exists a $\delta > 0$ such that

$$\left| f(s) - f(t) \right| < \beta \quad \text{if } x \in [a,b],\ t \in [a,b] \text{ and } \left| x - t \right| < \delta \quad \ldots\ldots\ldots(i)$$

If P is any partition of $[a,b]$ such that $\Delta x_i < \delta$ for all i

then (i) implies that $M_i - m_i \le \beta$, $(i = 1,2,\ldots.,n)$

$$\Rightarrow U\left(P,f,\alpha\right) - L\left(P,f,\alpha\right) = \sum M_i \,\Delta\alpha_i - \sum m_i\,\Delta\alpha_i$$
$$= \sum\left(M_i - m_i\right)\Delta\alpha_i$$
$$\le \beta \sum \Delta\alpha_i = \beta\left[\alpha(b)-\alpha(a)\right] < \varepsilon$$

$\Rightarrow$ $f \in \boldsymbol{R}(\alpha)$ by Cauchy Criterion. ⊙

> **Theorem**
> If f is monotonic on $[a,b]$, and if α is continuous on $[a,b]$, then $f \in \boldsymbol{R}(\alpha)$.
> (Monotonicity of α still assumed.)

Proof

Let $\varepsilon > 0$ be a given positive number.

For any positive integer n, choose a partition $P = \left\{x_0, x_1, \ldots.., x_n\right\}$ of $[a,b]$ such that

$$\Delta\alpha_i = \frac{\alpha(b)-\alpha(a)}{n} \quad , \quad i = 1,2,\ldots.,n$$

This is possible because α is continuous and monotonic increasing on the closed interval $[a,b]$ and thus assumes every value between its bounds, $\alpha(a)$ and $\alpha(b)$.

Let f be monotonic increasing on $[a,b]$, so that its lower and upper bounds m_i, M_i in $\left[x_{i-1}, x_i\right]$ are given by

$$m_i = f(x_{i-1}) \quad , \quad M_i = f(x_i) \quad , \quad i = 1,2,\ldots.,n$$

$$\therefore\ U\left(P,f,\alpha\right) - L\left(P,f,\alpha\right) = \sum_{i=1}^{n}\left(M_i - m_i\right)\Delta\alpha_i$$
$$= \frac{\alpha(b)-\alpha(a)}{n}\sum_{i=1}^{n}\left[f(x_i) - f(x_{i-1})\right]$$
$$= \frac{\alpha(b)-\alpha(a)}{n}\left[f(b) - f(a)\right]$$
$$< \varepsilon \qquad \text{if } n \text{ is taken large enough.}$$

$\Rightarrow$ $f \in R(\alpha)$ on $[a,b]$.

⊙

Note: $f \in \boldsymbol{R}(\alpha)$ when either

i) f is continuous and α is monotonic, or

ii) f is monotonic and α is continuous, of course α is still monotonic.

> **Properties of Integral**
>
> ***i***) If $f \in \boldsymbol{R}(\alpha)$ on $[a,b]$, then $cf \in \boldsymbol{R}(\alpha)$ for every constant c and
> $$\int_a^b cf\, d\alpha = c\int_a^b f\, d\alpha .$$

Proof

$\because\ f \in \boldsymbol{R}(\alpha)$

$\therefore\ \exists$ a partition P such that

$$U(P, f,\alpha) - L(P, f,\alpha) < \varepsilon \quad , \text{ where } \varepsilon \text{ is an arbitrary +ive number.}$$

Now $\quad U(P,cf,\alpha) = \sum_{i=1}^{n} cM_i\,\Delta\alpha_i = c\sum_{i=1}^{n} M_i\,\Delta\alpha_i$

$\&\quad L(P,cf,\alpha) = \sum_{i=1}^{n} cm_i\,\Delta\alpha_i = c\sum_{i=1}^{n} m_i\,\Delta\alpha_i$

$$\begin{aligned}\Rightarrow\ U(P, f,\alpha) - L(P, f,\alpha) &= c\left[\sum M_i\,\Delta\alpha_i - \sum m_i\,\Delta\alpha_i\right]\\ &= c\left[U(P, f,\alpha) - L(P, f,\alpha)\right]\\ &< c\varepsilon = \varepsilon_1\end{aligned}$$

$\Rightarrow\ cf \in \boldsymbol{R}(\alpha)$

$\because\ U(P,cf,\alpha) = c\left[U(P, f,\alpha)\right] \quad \& \quad L(P,cf,\alpha) = c\left[L(P, f,\alpha)\right]$

$\therefore\ \inf U(P,cf,\alpha) = c\left[\inf U(P, f,\alpha)\right] \ \&\ \sup L(P,cf,\alpha) = c\left[\sup L(P, f,\alpha)\right]$

where infimum and supremum are taken over all P on $[a,b]$.

$$\Rightarrow\ \int_a^{\bar{b}} cf\, d\alpha = c\int_a^{\bar{b}} f\, d\alpha \quad \& \quad \int_{\underline{a}}^{b} cf\, d\alpha = c\int_{\underline{a}}^{b} f\, d\alpha$$

$$\because\ \int_a^{\bar{b}} cf\, d\alpha = \int_{\underline{a}}^{b} cf\, d\alpha \quad \text{and} \quad \int_a^{\bar{b}} f\, d\alpha = \int_{\underline{a}}^{b} f\, d\alpha$$

$$\therefore\ \int_a^b cf\, d\alpha = c\int_a^b f\, d\alpha \qquad \odot$$

> ***ii***) If $f_1 \in \boldsymbol{R}(\alpha)$ and $f_2 \in \boldsymbol{R}(\alpha)$ on $[a,b]$, then $f_1 + f_2 \in \boldsymbol{R}(\alpha)$ and
> $$\int_a^b (f_1 + f_2)\, d\alpha = \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha .$$

Proof

If $f = f_1 + f_2$ and P is any partition of $[a,b]$, we have

$$m_i' + m_i'' \le m_i \le M_i \le M_i' + M_i''$$

where M_i', m_i', M_i'', m_i'' and M_i, m_i are the bounds of f_1, f_2 and f respectively in $[x_{i-1}, x_i]$.

Multiplying throughout by $\Delta\alpha_i$ and adding the inequalities for $i = 1, 2, \ldots, n$, we get

$$L(P, f_1,\alpha) + L(P, f_2,\alpha) \le L(P, f,\alpha) \le U(P, f,\alpha) \le U(P, f_1,\alpha) + U(P, f_2,\alpha) \ \ldots\ldots(i)$$

Since $f_1 \in \boldsymbol{R}(\alpha)$ and $f_2 \in \boldsymbol{R}(\alpha)$ on $[a,b]$ therefore $\exists\ \varepsilon > 0$ and there are partitions P_1 and P_2 such that

$$\left.\begin{aligned} U(P_1, f_1,\alpha) - L(P_1, f_1,\alpha) &< \varepsilon\\ \text{and}\quad U(P_2, f_2,\alpha) - L(P_2, f_2,\alpha) &< \varepsilon\end{aligned}\right\} \ \ldots\ldots\ldots\ldots (ii)$$

These inequalities hold if P_1 and P_2 are replaced by their common refinement P.

$(ii) \;\Rightarrow\; \left[U(P, f_1,\alpha)+U(P, f_2,\alpha)\right]-\left[L(P, f_1,\alpha)+L(P, f_2,\alpha)\right] < 2\varepsilon$

Using (i) we have

$$U(P, f,\alpha)-L(P, f,\alpha) < 2\varepsilon$$

which proves that $f \in \boldsymbol{R}(\alpha)$ on $[a,b]$

With the same partition P, we have

$$U(P, f_1,\alpha) < \int_a^b f_1\, d\alpha + \varepsilon$$

and $$U(P, f_2,\alpha) < \int_a^b f_2\, d\alpha + \varepsilon$$

Hence (i) implies that

$$\int_a^b f\, d\alpha \le U(P, f,\alpha) < \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha + 2\varepsilon$$

$\because$ ε is arbitrary, we conclude that

$$\int_a^b f\, d\alpha \le \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha$$

Similarly if we consider the lower sums we arrive at

$$\int_a^b f\, d\alpha \ge \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha$$

Combining the above two results, we have

$$\int_a^b f\, d\alpha = \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha$$ ⊙

iii) If $f_1(x) \le f_2(x)$ on $[a,b]$, then

$$\int_a^b f_1\, d\alpha \le \int_a^b f_2\, d\alpha$$

Proof

Let $f(x) \ge 0$, then $M_i \ge 0 \;\Rightarrow\; U(P, f,\alpha) \ge 0$

and $\therefore \int_a^b f\, d\alpha \ge 0$

$$\because\; f_1 \le f_2 \qquad \therefore\; f_2 - f_1 \ge 0$$

$$\Rightarrow \int_a^b (f_2 - f_1)\, d\alpha \ge 0 \qquad \Rightarrow \int_a^b f_2\, d\alpha - \int_a^b f_1\, d\alpha \ge 0$$

$$\Rightarrow \int_a^b f_1\, d\alpha \le \int_a^b f_2\, d\alpha$$ ⊙

➢ ***Note***

(i) $(f+g)(x) = f(x)+g(x) \le \sup f + \sup g$

$\Rightarrow \sup(f+g) \le \sup f + \sup g$

(ii) $(f+g)(x) = f(x)+g(x) \ge \inf f + \inf g$

$\Rightarrow \inf(f+g) \ge \inf f + \inf g$

iv) If $f \in \boldsymbol{R}(\alpha)$ on $[a,b]$ and if $a<c<b$, then $f \in \boldsymbol{R}(\alpha)$ on $[a,c]$ and on $[c,b]$ and

$$\int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha$$

Proof

Since $f \in \boldsymbol{R}(\alpha)$ on $[a,b]$, therefore for $\varepsilon>0$, $\exists$ a partition P such that

$$U(P,f,\alpha)-L(P,f,\alpha) < \varepsilon$$

Let P^* be the refinement of P such that $P^* = P\cup\{c\}$

$$\therefore\ L(P,f,\alpha) \le L(P^*,f,\alpha) \le U(P^*,f,\alpha) \le U(P,f,\alpha) \ \ldots\ldots\ldots\ldots (i)$$

$$\Rightarrow\ U(P^*,f,\alpha)-L(P^*,f,\alpha) \le U(P,f,\alpha)-L(P,f,\alpha) < \varepsilon \ \ldots\ldots\ldots (ii)$$

Let P_1, P_2 denote the sets of points of P^* between $[a,c]$, $[c,b]$ respectively.

Clearly P_1, P_2 are partitions of $[a,c]$, $[c,b]$ respectively and $P^* = P_1 \cup P_2$.

Also $\quad U(P^*,f,\alpha)=U(P_1,f,\alpha)+U(P_2,f,\alpha) \ \ldots\ldots\ldots (iii)$

and $\quad L(P^*,f,\alpha)=L(P_1,f,\alpha)+L(P_2,f,\alpha) \ \ldots\ldots\ldots\ldots (iv)$

$$\therefore\ \{U(P_1,f,\alpha)-L(P_1,f,\alpha)\}+\{U(P_2,f,\alpha)-L(P_2,f,\alpha)\}$$
$$=U(P^*,f,\alpha)-L(P^*,f,\alpha)<\varepsilon$$

Since each bracket on the left is non-negative, it follows that

$$U(P_1,f,\alpha)-L(P_1,f,\alpha) < \varepsilon$$

and $\quad U(P_2,f,\alpha)-L(P_2,f,\alpha) < \varepsilon$

$\Rightarrow\ f \in \boldsymbol{R}(\alpha)$ on $[a,c]$ and on $[c,b]$.

We know that for any functions f_1 and f_2, if $f = f_1 + f_2$, then

$$\inf f \ \ge\ \inf f_1 + \inf f_2$$

and $\quad \sup f \ \le\ \sup f_1 + \sup f_2$

Now for any partitions P_1, P_2 of $[a,c]$, $[c,b]$ respectively, if $P^* = P_1 \cup P_2$, then

$$U(P^*,f,\alpha)=U(P_1,f,\alpha)+U(P_2,f,\alpha)$$

Hence on taking the infimum for all partitions, we get

$$\overline{\int_a^b} f\,d\alpha \ \ge\ \overline{\int_a^c} f\,d\alpha + \overline{\int_c^b} f\,d\alpha$$

But since $f \in \boldsymbol{R}(\alpha)$ on $[a,c]$, $[c,b]$, $[a,b]$

$$\therefore\ \int_a^b f\,d\alpha \ \ge\ \int_a^c f\,d\alpha + \int_c^b f\,d\alpha \ \ldots\ldots\ldots\ldots (v)$$

Again $\quad L(P^*,f,\alpha)=L(P_1,f,\alpha)+L(P_2,f,\alpha)$

and on taking the supremum for all partitions, we get

$$\underline{\int_a^b} f\,d\alpha \ \le\ \underline{\int_a^c} f\,d\alpha + \underline{\int_c^b} f\,d\alpha$$

But since $f \in \boldsymbol{R}(\alpha)$ on $[a,c]$, $[c,b]$, $[a,b]$

$$\therefore\ \int_a^b f\,d\alpha \ \le\ \int_a^c f\,d\alpha + \int_c^b f\,d\alpha \ \ldots\ldots\ldots\ldots\ldots (vi)$$

(v) and (vi) imply that

$$\int_a^b f\,d\alpha \ = \ \int_a^c f\,d\alpha + \int_c^b f\,d\alpha \qquad \odot$$

v) If $f \in \boldsymbol{R}(\alpha)$ on $[a,b]$ and $\left| f(x) \right| \le M$ on $[a,b]$, then

$$\left| \int_a^b f \, d\alpha \right| \le M\left[\alpha(b) - \alpha(a)\right]$$

Proof

We know that

$$\int_a^b f \, d\alpha \le U(P, f, \alpha)$$

$$= \sum M_i \, \Delta\alpha_i \le M \sum \Delta\alpha_i$$

But

$$\sum \Delta\alpha_i = \alpha(b) - \alpha(a)$$

$$\Rightarrow \left| \int_a^b f \, d\alpha \right| \le M\left[\alpha(b) - \alpha(a)\right]$$

⊙

vi) If $f \in \boldsymbol{R}(\alpha_1)$ and $f \in \boldsymbol{R}(\alpha_2)$, then $f \in \boldsymbol{R}(\alpha_1 + \alpha_2)$ and

$$\int_a^b f \, d(\alpha_1 + \alpha_2) = \int_a^b f \, d\alpha_1 + \int_a^b f \, d\alpha_2$$

and if $f \in \boldsymbol{R}(\alpha)$ and c is a positive constant, then $f \in \boldsymbol{R}(c\alpha)$ and

$$\int_a^b f \, d(c\alpha) = c \int_a^b f \, d\alpha$$

Proof

Since $f \in \boldsymbol{R}(\alpha_1)$ and $f \in \boldsymbol{R}(\alpha_2)$, therefore for $\varepsilon > 0$, there exists partitions P_1, P_2 of $[a,b]$ such that

$$U(P_1, f, \alpha_1) - L(P_1, f, \alpha_1) < \frac{\varepsilon}{2}$$

$$\text{and} \quad U(P_2, f, \alpha_2) - L(P_2, f, \alpha_2) < \frac{\varepsilon}{2}$$

Let $P = P_1 \cup P_2$

$$\left.\begin{aligned} \therefore \; & U(P, f, \alpha_1) - L(P, f, \alpha_1) < \frac{\varepsilon}{2} \\ \& \; & U(P, f, \alpha_2) - L(P, f, \alpha_2) < \frac{\varepsilon}{2} \end{aligned}\right\} \; \ldots\ldots\ldots\ldots.. (i)$$

Let m_i , M_i be bounds of f in $\left[x_{i-1}, x_i\right]$

Take $\alpha = \alpha_1 + \alpha_2$

$$\Rightarrow \Delta\alpha_i = \Delta\alpha_{1i} + \Delta\alpha_{2i}$$

$$\begin{aligned} \therefore \; U(P, f, \alpha) &= \sum M_i \, \Delta\alpha_i \\ &= \sum M_i \left(\Delta\alpha_{1i} + \Delta\alpha_{2i}\right) \\ &= U(P, f, \alpha_1) + U(P, f, \alpha_2) \end{aligned}$$

Similarly

$$L(P, f, \alpha) = L(P, f, \alpha_1) + L(P, f, \alpha_2)$$

$$\therefore \; U(P, f, \alpha) - L(P, f, \alpha) = U(P, f, \alpha_1) - L(P, f, \alpha_1) + U(P, f, \alpha_2) - L(P, f, \alpha_2)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \text{by } (i)$$

$$\Rightarrow f \in \boldsymbol{R}(\alpha) \quad \text{where} \quad \alpha = \alpha_1 + \alpha_2$$

To prove the second part, we notice that

$$\begin{aligned}\int_a^b f\,d\alpha &= \inf U(P,f,\alpha)\\ &= \inf\{U(P,f,\alpha_1)+U(P,f,\alpha_2)\}\\ &\geq \inf U(P,f,\alpha_1)+\inf U(P,f,\alpha_2)\\ &= \int_a^b f\,d\alpha_1+\int_a^b f\,d\alpha_2 \quad \ldots\ldots\ldots\ldots\ldots (ii)\end{aligned}$$

Similarly by taking the supremum of lower sum of partition we arrive that

$$\int_a^b f\,d\alpha \leq \int_a^b f\,d\alpha_1+\int_a^b f\,d\alpha_2 \quad \ldots\ldots\ldots\ldots\ldots (iii)$$

From (*ii*) and (*iii*)

$$\int_a^b f\,d\alpha = \int_a^b f\,d\alpha_1+\int_a^b f\,d\alpha_2$$

i.e. $$\int_a^b f\,d(\alpha_1+\alpha_2) = \int_a^b f\,d\alpha_1+\int_a^b f\,d\alpha_2 \qquad \because\ \alpha=\alpha_1+\alpha_2$$

Now $\because$ $f\in \boldsymbol{R}(\alpha)$ $\therefore$ for $\varepsilon>0$, $\exists$ a partition P of $[a,b]$ such that

$$U(P,f,\alpha)-L(P,f,\alpha) < \varepsilon \quad \ldots\ldots\ldots\ldots (iv)$$

Let $\alpha'=c\alpha$ then $\Delta\alpha_i'=\Delta(c\alpha_i)=c\Delta\alpha_i$

$$\begin{aligned}\Rightarrow\ U(P,f,\alpha') &= \sum M_i\,\Delta\alpha_i'\\ &= \sum M_i\,(c\Delta\alpha_i)\\ &= c\sum M_i\,\Delta\alpha_i\\ &= c\,U(P,f,\alpha)\end{aligned}$$

Similarly, $L(P,f,\alpha')=c\,L(P,f,\alpha)$

$$\Rightarrow\ U(P,f,\alpha')-L(P,f,\alpha')=c\{U(P,f,\alpha)-L(P,f,\alpha)\}<c\varepsilon \qquad \text{by } (iv)$$

$$\Rightarrow\ f\in \boldsymbol{R}(\alpha') \quad \text{where } \alpha'=c\alpha$$

Also
$$\begin{aligned}\int_a^b f\,d\alpha' &= \inf U(P,f,\alpha')\\ &= \inf\ c\,U(P,f,\alpha)\\ &= c\inf U(P,f,\alpha)\\ &= c\int_a^b f\,d\alpha\end{aligned}$$

and

$$\begin{aligned}\int_a^b f\,d\alpha' &= \sup L(P,f,\alpha')\\ &= \sup cU(P,f,\alpha)\\ &= c\sup U(P,f,\alpha)\\ &= c\int_a^b f\,d\alpha\end{aligned}$$

Hence

$$\int_a^b f\,d\alpha' = c\int_a^b f\,d\alpha \quad \text{where } \alpha'=c\alpha$$

⊙

> ### ➢ *Lemma*
> If M & m are the supremum and infimum of f and M', m' are the supremum & infimum of $|f|$ on $[a,b]$ then $M'-m' \leq M-m$.

Proof

Let $x_1, x_2 \in [a,b]$, then

$$\big|\,|f(x_1)|-|f(x_2)|\,\big| \leq |f(x_1)-f(x_2)| \quad \ldots\ldots\ldots\ldots\ldots\ldots(A)$$

$\because$ M and m denote the supremum and infimum of $f(x)$ on $[a,b]$

$\therefore$ $f(x) \leq M$ & $f(x) \geq m \quad \forall\ x \in [a,b]$

$\because$ $x_1, x_2 \in [a,b]$

$\therefore$ $f(x_1) \leq M$ and $f(x_2) \geq m$

$\Rightarrow$ $f(x_1) \leq M$ and $-f(x_2) \leq -m$

$\Rightarrow$ $f(x_1)-f(x_2) \leq M-m$ ……………... (i)

Interchanging x_1 & x_2, we get

$$-\big[f(x_1)-f(x_2)\big] \leq M-m \quad \ldots\ldots\ldots\ldots\ (ii)$$

(i) & (ii) $\Rightarrow$ $|f(x_1)-f(x_2)| \leq M-m$

$$\Rightarrow \big|\,|f(x_1)|-|f(x_2)|\,\big| \leq M-m \quad \text{by eq. } (A) \ldots\ldots\ldots\ldots(\mathrm{I})$$

$\because$ M' and m' denote the supremum and infimum of $|f(x)|$ on $[a,b]$

$\therefore$ $|f(x)| \leq M'$ and $|f(x)| \geq m' \quad \forall\ x \in [a,b]$

$\Rightarrow$ $\exists\ \varepsilon > 0$ such that

$$|f(x_1)| > M'-\varepsilon \quad \ldots\ldots\ldots\ldots\ (iii)$$

$$\text{and} \quad |f(x_2)| < m'+\varepsilon \quad \Rightarrow \quad -|f(x_2)|+\varepsilon > -m' \quad \ldots\ldots\ldots\ldots\ (iv)$$

From (iii) and (iv), we get

$$|f(x_1)|-|f(x_2)|+\varepsilon > M'-m'-\varepsilon$$

$$\Rightarrow 2\varepsilon+|f(x_1)|-|f(x_2)| > M'-m'$$

$\because$ ε is arbitrary $\therefore$ $M'-m' \leq |f(x_1)|-|f(x_2)|$ ……….…. (v)

Interchanging x_1 & x_2, we get

$$M'-m' \leq -\big(|f(x_1)|-|f(x_2)|\big) \quad \ldots\ldots\ldots\ldots\ldots\ (vi)$$

Combining (v) and (vi), we get

$$M'-m' \leq \big|\,|f(x_1)|-|f(x_2)|\,\big| \quad \ldots\ldots\ldots\ldots\ldots\ (\mathrm{II})$$

From (I) and (II), we have the require result

$$M'-m' \leq M-m$$

⊙

> ### ➢ *Theorem*
> If $f \in \boldsymbol{R}(\alpha)$ on $[a,b]$, then $|f| \in \boldsymbol{R}(\alpha)$ on $[a,b]$ and $\left|\int_a^b f\,d\alpha\right| \leq \int_a^b |f|\,d\alpha$.

Proof

$\because$ $f \in \boldsymbol{R}(\alpha)$

$\therefore$ given $\varepsilon > 0$ $\exists$ a partition P of $[a,b]$ such that

$$U(P,f,\alpha)-L(P,f,\alpha) < \varepsilon$$

$$\text{i.e.} \quad \sum M_i\,\Delta\alpha_i-\sum m_i\,\Delta\alpha_i=\sum\big(M_i-m_i\big)\Delta\alpha_i < \varepsilon$$

Where M_i and m_i are supremum and infimum of f on $\big[x_{i-1}, x_i\big]$

Now if M_i' and m_i' are supremum and infimum of $|f|$ on $\big[x_{i-1}, x_i\big]$ then

$$M_i'-m_i' \leq M_i-m_i$$

$$\Rightarrow \sum\left(M_i' - m_i'\right)\Delta\alpha_i \le \sum\left(M_i - m_i\right)\Delta\alpha_i$$
$$\Rightarrow U\left(P,|f|,\alpha\right) - L\left(P,|f|,\alpha\right) \le U\left(P,f,\alpha\right) - L\left(P,f,\alpha\right) < \varepsilon$$
$$\Rightarrow |f| \in \boldsymbol{R}(\alpha).$$

Take $c = +1$ or -1 to make $c\int f\,d\alpha \ge 0$

Then $\left|\int_a^b f\,d\alpha\right| = c\int_a^b f\,d\alpha$ .... (i)

Also $c\,f(x) \le |f(x)| \quad \forall\ x \in [a,b]$

$$\Rightarrow \int_a^b c\,f\,d\alpha \le \int_a^b |f|\,d\alpha \Rightarrow c\int_a^b f\,d\alpha \le \int_a^b |f|\,d\alpha \text{ .... } (ii)$$

From (i) and (ii), we have

$$\left|\int_a^b f\,d\alpha\right| \le \int_a^b |f|\,d\alpha$$ ⊙

> ➢ ***Theorem***
> If $f \in \boldsymbol{R}(\alpha)$ on $[a,b]$, then $f^2 \in \boldsymbol{R}(\alpha)$ on $[a,b]$.

Proof

$\because\ f \in \boldsymbol{R}(\alpha) \quad \Rightarrow |f| \in \boldsymbol{R}(\alpha)$

$\Rightarrow |f(x)| < M \quad \forall\ x \in [a,b]$

$\because\ f \in \boldsymbol{R}(\alpha) \quad \therefore$ given $\varepsilon > 0$, $\exists$ a partition P of $[a,b]$ such that

$$U\left(P,f,\alpha\right) - L\left(P,f,\alpha\right) < \varepsilon/2M \text{ .... } (i)$$

If M_i & m_i denote the sup. & inf. of f on $\left[x_{i-1},x_i\right]$ then M_i^2 & m_i^2 are the sup. & inf. of f^2 on $\left[x_{i-1},x_i\right]$.

$$\Rightarrow U\left(P,f^2,\alpha\right) - L\left(P,f^2,\alpha\right) = \sum\left(M_i^2 - m_i^2\right)\Delta\alpha_i$$
$$= \sum\left(M_i + m_i\right)\left(M_i - m_i\right)\Delta\alpha_i$$

$\because\ f(x) \le |f(x)| \le M \quad \forall\ x \in [a,b]$

and $f^2 = |f|^2$

$\therefore\ M_i \le M$ & $m_i \le M$

$$\Rightarrow U\left(P,f^2,\alpha\right) - L\left(P,f^2,\alpha\right) \le \sum\left(M + M\right)\left(M_i - m_i\right)\Delta\alpha_i$$
$$= 2M\sum\left(M_i - m_i\right)\Delta\alpha_i$$
$$= 2M\left[U\left(P,f,\alpha\right) - L\left(P,f,\alpha\right)\right] < 2M \cdot \frac{\varepsilon}{2M} = \varepsilon$$

$\Rightarrow f^2 \in \boldsymbol{R}(\alpha)$ ⊙

> ➢ ***Corollary***
> If $f \in \boldsymbol{R}(\alpha)$ & $g \in \boldsymbol{R}(\alpha)$ on $[a,b]$ then $fg \in \boldsymbol{R}(\alpha)$ on $[a,b]$.

Proof

$\because\ f \in \boldsymbol{R}(\alpha)$, $g \in \boldsymbol{R}(\alpha)$

$\therefore\ f + g \in \boldsymbol{R}(\alpha)$, $f - g \in \boldsymbol{R}(\alpha)$

$\Rightarrow (f+g)^2 \in \boldsymbol{R}(\alpha)$, $(f-g)^2 \in \boldsymbol{R}(\alpha)$

$\Rightarrow (f+g)^2 - (f-g)^2 \in \boldsymbol{R}(\alpha) \Rightarrow 4fg \in \boldsymbol{R}(\alpha)$

and ultimately

$$fg \in \boldsymbol{R}(\alpha) \text{ on } [a,b]$$ ⊙

➢ ***Theorem***

Assume α increases monotonically and $\alpha' \in \boldsymbol{R}$ on $[a,b]$. Let f be bounded real function on $[a,b]$. Then $f \in \boldsymbol{R}(\alpha)$ iff $f\alpha' \in \boldsymbol{R}$. In that case

$$\int_a^b f\,d\alpha = \int_a^b f(x)\cdot\alpha'(x)\,dx$$

Proof

$\because\ \alpha' \in \boldsymbol{R}$ on $[a,b]$

$\therefore$ given $\varepsilon > 0\ \exists$ a partition P of $[a,b]$ such that

$$U(P,\alpha') - L(P,\alpha') < \varepsilon \quad \ldots\ldots\ldots\ldots\ (i)$$

The Mean-value theorem furnishes point $t_i \in [x_{i-1}, x_i]$ such that

$$\begin{aligned}\Delta\alpha_i &= \alpha(x_i) - \alpha(x_{i-1}) \\ &= \alpha'(t_i)\Delta x_i \quad \text{for } i = 1,2,\ldots,n \quad \ldots\ldots\ldots\ (ii)\end{aligned}$$

If $s_i \in [x_{i-1}, x_i]$, then form (i) we have

$$\left|\sum \alpha'(s_i)\Delta x_i - \sum \alpha'(t_i)\Delta x_i\right| < \varepsilon \qquad \text{| Previously proved at page 6}$$

$$\Rightarrow\ \sum \left|\alpha'(s_i) - \alpha'(t_i)\right|\Delta x_i < \varepsilon \quad \ldots\ldots\ldots\ldots\ (iii)$$

Put $M = \sup\left|f(x)\right|$ and consider

$$\begin{aligned}&\left|\sum f(s_i)\Delta\alpha_i - \sum f(s_i)\alpha'(s_i)\Delta x_i\right| \quad \ldots\ldots\ldots\ldots\ (A)\\ &= \left|\sum f(s_i)\alpha'(t_i)\Delta x_i - \sum f(s_i)\alpha'(s_i)\Delta x_i\right| \qquad \text{by } (ii)\\ &= \left|\sum f(s_i)\big(\alpha'(t_i) - \alpha'(s_i)\big)\Delta x_i\right|\\ &\le \left|\sum M\big(\alpha'(t_i) - \alpha'(s_i)\big)\right|\Delta x_i\\ &\le M\varepsilon \qquad \ldots\ldots\ldots\ldots\ (iv) \qquad \text{by } (iii)\end{aligned}$$

$\Rightarrow\ \sum f(s_i)\Delta\alpha_i \le \sum f(s_i)\alpha'(s_i)\Delta x_i + M\varepsilon$ for all choices of $s_i \in [x_{i-1}, x_i]$

$\Rightarrow\ U(P, f, \alpha) \le U(P, f\alpha') + M\varepsilon$

The same arguments leads from (A) to

$$U(P, f\alpha') \le U(P, f, \alpha) + M\varepsilon$$

Thus $\left|U(P, f, \alpha) - U(P, f\alpha')\right| \le M\varepsilon \quad \ldots\ldots\ldots\ldots\ (v)$

$\because$ (i) remains true if P is replaced by any refinement

$\therefore$ (v) also remains true

$$\Rightarrow\ \left|\overline{\int_a^b} f\,d\alpha - \overline{\int_a^b} f(x)\alpha'(x)\,dx\right| \le M\varepsilon$$

$\because\ \varepsilon$ was arbitrary

$$\therefore\ \overline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f(x)\alpha'(x)\,dx \qquad \text{for any bounded } f.$$

Using the same argument, we can prove from (iv) by considering the infimum of $\left|f(x)\right|$ that

$$\underline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f(x)\alpha'(x)\,dx$$

Hence

$$\overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha \iff \overline{\int_a^b} f(x)\alpha'(x)\,dx = \underline{\int_a^b} f(x)\alpha'(x)\,dx$$

Equivalently $f \in \boldsymbol{R}(\alpha) \iff f\alpha' \in \boldsymbol{R}(\alpha)$. ⊙

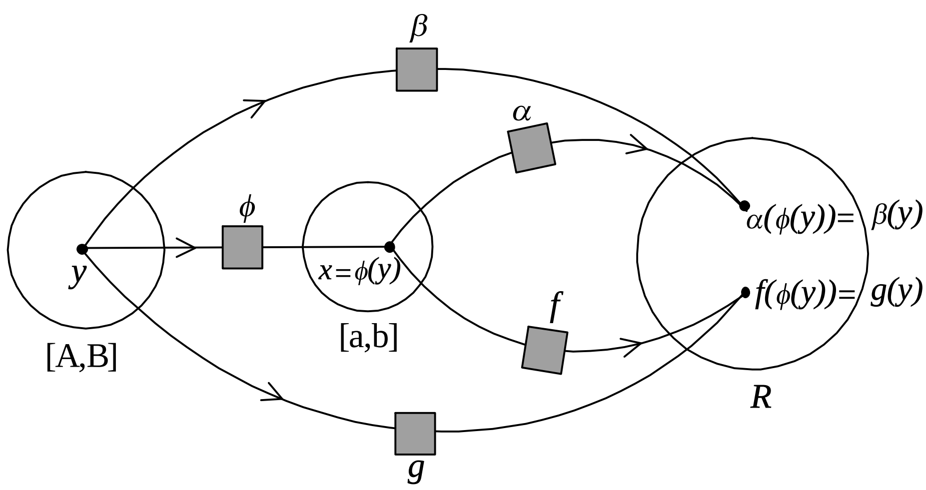
➢ *Theorem (Change of Variable)*

Suppose φ is a strictly increasing continuous function that maps an interval $[A,B]$ onto $[a,b]$. Suppose α is monotonically increasing on $[a,b]$ and $f \in \boldsymbol{R}(\alpha)$ on $[a,b]$. Define β and g on $[A,B]$ by

$$\beta(y) = \alpha(\varphi(y)) \quad , \quad g(y) = f(\varphi(y))$$

then $g \in \boldsymbol{R}(\beta)$ and $\int_A^B g\,d\beta = \int_a^b f\,d\alpha$.

Proof



To each partition $P = \{x_0, \ldots\ldots, x_n\}$ of $[a,b]$ corresponds a partition $Q = \{y_0, \ldots\ldots, y_n\}$ of $[A,B]$ because φ maps $[A,B]$ onto $[a,b]$.

$\Rightarrow \; x_i = \varphi(y_i)$

All partitions of $[A,B]$ are obtained in this way.

$\because$ The value taken by f on $[x_{i-1}, x_i]$ are exactly the same as those taken by g on $[y_{i-1}, y_i]$, we see that

$$U(Q,g,\beta) = U(P,f,\alpha)$$

$$\text{and} \quad L(Q,g,\beta) = L(P,f,\alpha)$$

$\because \; f \in R(\alpha)$ on $[a,b]$

$\therefore$ given $\varepsilon > 0$, we have

$$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$$

$$\Rightarrow U(Q,g,\beta) - L(Q,g,\beta) < \varepsilon$$

$\Rightarrow \; g \in \boldsymbol{R}(\beta)$ and $\int_A^B g\,d\beta = \int_a^b f\,d\alpha$ ⊙

INTEGRATION AND DIFFERENTIATION

> ➢ ***Theorem (Ist Fundamental Theorem of Calculus)***
>
> Let $f \in \boldsymbol{R}$ on $[a,b]$. For $a \le x \le b$, put $F(x) = \int_a^x f(t)\,dt$, then F is continuous on $[a,b]$; furthermore, if f is continuous at point x_0 of $[a,b]$, then F is differentiable at x_0, and $F'(x_0) = f(x_0)$.

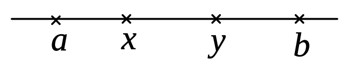
Proof

$\because\ f \in \boldsymbol{R}$

$\therefore\ f$ is bounded.

Let $|f(t)| \le M$ for $t \in [a,b]$

If $a \le x < y \le b$, then



$$|F(y) - F(x)| = \left| \int_a^y f(t)\,dt - \int_a^x f(t)\,dt \right|$$

$$= \left| \int_a^x f(t)\,dt + \int_x^y f(t)\,dt - \int_a^x f(t)\,dt \right|$$

$$= \left| \int_x^y f(t)\,dt \right| \le \int_x^y |f(t)|\,dt \le M \int_x^y dt = M(y-x)$$

$$\Rightarrow |F(y) - F(x)| < \varepsilon \quad \text{for } \varepsilon > 0 \text{ provided } M|y-x| < \varepsilon$$

i.e. $|F(y) - F(x)| < \varepsilon$ whenever $|y - x| < \dfrac{\varepsilon}{M}$

This proves the continuity (and, in fact, uniform continuity) of F on $[a,b]$.

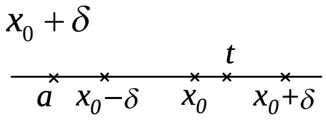
Next, we have to prove that if f is continuous at $x_0 \in [a,b]$ then F is differentiable at x_0 and $F'(x_0) = f(x_0)$

$$\text{i.e.} \quad \lim_{t \to x_0} \frac{F(t) - F(x_0)}{t - x_0} = f(x_0)$$

Suppose f is continuous at x_0. Given $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$$|f(t) - f(x_0)| < \varepsilon \quad \text{if} \quad |t - x_0| < \delta \quad \text{where } t \in [a,b]$$

$$\Rightarrow f(x_0) - \varepsilon < f(t) < f(x_0) + \varepsilon \quad \text{if} \quad x_0 - \delta < t < x_0 + \delta$$



$$\Rightarrow \int_{x_0}^t \big(f(x_0) - \varepsilon\big)dt < \int_{x_0}^t f(t)\,dt < \int_{x_0}^t \big(f(x_0) + \varepsilon\big)dt$$

$$\Rightarrow \big(f(x_0) - \varepsilon\big)\int_{x_0}^t dt < \int_{x_0}^t f(t)\,dt < \big(f(x_0) + \varepsilon\big)\int_{x_0}^t dt$$

$$\Rightarrow \big(f(x_0) - \varepsilon\big)(t - x_0) < F(t) - F(x_0) < \big(f(x_0) + \varepsilon\big)(t - x_0)$$

$$\Rightarrow f(x_0) - \varepsilon < \frac{F(t) - F(x_0)}{t - x_0} < f(x_0) + \varepsilon$$

$$\Rightarrow \left| \frac{F(t) - F(x_0)}{t - x_0} - f(x_0) \right| < \varepsilon$$

$$\Rightarrow \lim_{t \to x_0} \frac{F(t) - F(x_0)}{t - x_0} = f(x_0)$$

$$\Rightarrow F'(x_0) = f(x_0)$$

⊙

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➢ Theorem (IInd Fundamental Theorem of Calculus)

If $f \in \boldsymbol{R}$ on $[a,b]$ and if there is a differentiable function F on $[a,b]$ such that $F' = f$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Proof

$\because$ $f \in \boldsymbol{R}$ on $[a,b]$

$\therefore$ given $\varepsilon > 0$, $\exists$ a partition P of $[a,b]$ such that

$$U(P,f) - L(P,f) < \varepsilon$$

$\because$ F is differentiable on $[a,b]$

$\therefore$ $\exists$ $t_i \in [x_{i-1}, x_i]$ such that

$$F(x_i) - F(x_{i-1}) = F'(t_i)\Delta x_i$$

$$\Rightarrow F(x_i) - F(x_{i-1}) = f(t_i)\Delta x_i \quad \text{for } i = 1,2,....,n \quad \because F' = f$$

$$\Rightarrow \sum_{i=1}^{n} f(t_i)\Delta x_i = F(b) - F(a)$$

$$\Rightarrow \left| F(b) - F(a) - \int_a^b f(x)\,dx \right| < \varepsilon$$

$\because$ if $f \in \boldsymbol{R}(\alpha)$ then

$$\left| \sum f(t_i)\Delta\alpha_i - \int_a^b f\,d\alpha \right| < \varepsilon$$

$\because$ ε is arbitrary

$\therefore$ $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$ ⊙

➢ Theorem (Integration by Parts)

Suppose F and G are differentiable function on $[a,b]$, $F' = f \in \boldsymbol{R}$ and $G' = g \in \boldsymbol{R}$ then

$$\int_a^b F(x)g(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)\,dx$$

Proof

Put $H(x) = F(x)G(x)$

$\Rightarrow$ $H' = F'(x)G(x) + F(x)G'(x) = h$

Now $\because$ $H \in \boldsymbol{R}$ and $h \in \boldsymbol{R}$ on $[a,b]$

$\therefore$ By applying the fundamental theorem of calculus to H and its derivative h, we have

$$\int_a^b h\,dx = H(b) - H(a)$$

$$\Rightarrow \int_a^b \left[F'(x)G(x) + F(x)G'(x)\right]dx = H(b) - H(a)$$

$$\Rightarrow \int_a^b f(x)G(x)\,dx + \int_a^b F(x)g(x)\,dx = F(b)G(b) - F(a)G(a)$$

$$\Rightarrow \int_a^b F(x)g(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)\,dx$$ ⊙

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> **Question**
>
> Show that the function f defined on $[0,1]$ by
> $$f(x)=\begin{cases}1 & ;\ x \text{ is rational}\\ 0 & ;\ x \text{ is irrational}\end{cases}$$
> is not integrable on $[0,1]$

Solution

For any partition P of $[0,1]$, $m_k=0$, $M_k=1$

$$\Rightarrow S(P,f)=\sum_{k=1}^{n}M_k\,\Delta x_k=\sum_{k=1}^{n}\Delta x_k=1-0=1$$

and $$L(P,f)=\sum_{k=1}^{n}m_k\,\Delta x_k=0$$

so that $\overline{\int_0^1} f\,dx=1$, $\underline{\int_0^1} f\,dx=0$

i.e. $\overline{\int_0^1} f\,dx\neq\underline{\int_0^1} f\,dx \quad\Rightarrow f$ is not integrable on $[0,1]$. ⊙

> **Question**
>
> Show that $f(x)=\sin x$ is Riemann integrable over $\left[0,\frac{\pi}{2}\right]$.

Solution

Take $P=\left\{0,\frac{\pi}{2n},\frac{\pi}{n},\frac{3\pi}{2n},\ldots\ldots,\frac{n\pi}{2n}\right\}$ by dividing $\left[0,\frac{\pi}{2}\right]$ into n equal parts.

Then $M_k=\sin\frac{k\pi}{2n}$, $m_k=\sin\frac{(k-1)\pi}{2n}$

$$\Rightarrow S(P,f)-L(P,f)=\sum\left(\sin\frac{k\pi}{2n}-\sin\frac{(k-1)\pi}{2n}\right)\frac{\pi}{2n}$$

$$\leq\frac{\pi}{2n}<\varepsilon \qquad \text{for } n>n_0=\frac{\pi}{2\varepsilon}$$

$\Rightarrow f$ is Riemann integrable over $\left[0,\frac{\pi}{2}\right]$. ⊙

> **Question**
>
> Show that $f(x)=\begin{cases}1/x & ;\ x \text{ is rational}, 0<x\leq 1\\ 0 & ;\ x \text{ is irrational}\end{cases}$
>
> is integrable on $[0,1]$.

Solution

f is continuous at each irrational. And rational numbers are dense in $[0,1]$.

Also $L(P,f)=0$ for any partition P of $[0,1]$ so that $\underline{\int_0^1} f\,dx=0$

$\because f\geq 0 \quad \therefore S(P,f)\geq 0 \qquad \Rightarrow \overline{\int_0^1} f\,d\alpha\geq 0$ ………… (i)

$\because$ There are only finite number of points $\dfrac{p}{q}$ (rationals) for which $f\left(\dfrac{p}{q}\right)=\dfrac{q}{p}\geq\dfrac{\varepsilon}{2}$

$\therefore$ Suppose $f(x)\geq\dfrac{\varepsilon}{2}$ for k values of x in $[0,1]$

Take P_1 such that $|P_1|<\dfrac{\varepsilon}{2k}$.

Consider $S(P_1,f)=\sum_{i=1}^{n}M_i(x_i-x_{i-1})$

There are at most k values for which $\dfrac{\varepsilon}{2}\leq M_i\leq 1$. For all other values $M_i>\dfrac{\varepsilon}{2}$.

$$\Rightarrow S(P_1,f)=\sum_{k\text{ values}}M_i(x_i-x_{i-1})+\sum_{\textit{other}\text{ values}}M_i(x_i-x_{i-1})$$

$$\leq\frac{\varepsilon}{2k}\cdot k+\frac{\varepsilon}{2}\sum(x_i-x_{i-1})<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

$\because$ ε is arbitrary

$\therefore$ $S(P_1,f)\leq 0$ and $\overline{\int_0^1} f\,dx\leq 0$ ………… (*ii*)

By (*i*) and (*ii*), we have

$$\overline{\int_0^1} f\,dx=0$$

Hence $\int_0^1 f\,dx=0$ ⊙

➢ *Note*

If f is integrable then $|f|$ is also integrable but the converse is false.

For example, let f be a function defined on $[a,b]$ by

$$f(x)=\begin{cases}1 & ;\ x\in\mathbb{Q}\cap[a,b]\\ -1 & ;\ \text{otherwise}\end{cases}$$

Then $|f|$ is Riemann-integrable but f is not.